Bayesian VAR in gretl

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Introduction

Bayesian VAR models have become a standard practice in modern macroeconometrics.

$\downarrow$

The introduction of **priors** impose structure to parameters:

- **shrinkage** is induced to over-parametrized context;
- **more accurate** predictions/analysis ("short" sample).

The package

The `BVAR` package (joint with Sven Schreiber) provides Bayesian VAR routines to the `gretl` ecosystem.

Main features:

- many prior specifications such as fixed Σ, conjugate, independent, large set-up (Bańbura et al., 2010);
- posterior simulation via Gibbs sampling with MPI;
- posterior predictions;
- posterior impulse response functions (Cholesky)

The workflow

1. Initialization → `BVAR_setup();`
2. Prior customization → `BVAR_alpha_prior()`, `BVAR_sigma_prior();`
3. Posterior computation →`BVAR_posterior();`
4. Print/Plot → `BVAR_printout()`, `BVAR_fcast_plot()`, `BVAR_irf_plot()`.

VAR framework

Let us define a VAR(p) model as:

$$\underset{1\times m}{y_t} = a_0 + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma)$$

which can be rewritten into:

- *matric-variate form* $\underset{T\times m}{Y} = XA + E;$
- *multivariate (vec) form* $\underset{Tm\times 1}{y} = (I_m \otimes X)\alpha + \varepsilon.$

Likelihood

Bayesian analysis revolves around **posterior** distributions

$$\underbrace{P(\theta|y)}_{posterior} \propto \underbrace{p(y|\theta)}_{likelihood} \underbrace{P(\theta)}_{prior}, \qquad \theta = (A, \Sigma)$$

The likelihood is easily derived as follows:

$$p(y|A,\Sigma) \propto |\Sigma|^{-T/2} \exp\left[-\frac{1}{2} tr\big((Y - XA)'(Y - XA)\Sigma^{-1}\big)\right]$$

Similarly to the standard linear model,

$$p(y|\alpha,\Sigma) \propto |\Sigma|^{-\frac{1+mp}{2}} \exp\left[-\frac{1}{2}tr\big((\hat{A}-A)'X'X(\hat{A}-A)\Sigma^{-1}\big)\right]\times$$
$$|\Sigma|^{-\frac{T-(1+mp)}{2}} \exp\left[-\frac{1}{2}tr\big((Y-X\hat{A})'(Y-X\hat{A})\Sigma^{-1}\big)\right]$$

where $\hat{A}=(X'X)^{-1}X'y$ is the OLS estimator.

The likelihood is made by a Multivariate Normal kernel and an Inverse Wishart one.

Fixed Σ case

Historically, the first BVAR formulation assumed Σ as deterministic

$$P(\alpha, \Sigma) = P(\alpha)$$

The posterior for α,

$$\alpha | y \sim N(\tilde{\alpha}, \tilde{V})$$

where

$$\tilde{V} = [\underline{V}^{-1} + (\Sigma^{-1} \otimes (X'X))]^{-1}$$
$$\tilde{\alpha} = \tilde{V}[\underline{V}^{-1}\underline{\alpha} + (\Sigma^{-1} \otimes X)'y]$$

The Minnesota prior (Litterman, 1986)

$$\alpha \sim \mathrm{MVN}(\underline{\alpha}, \underline{V})$$

- $\underline{\alpha}$ has its *r-th* element

$$\underline{\alpha}_r = \begin{cases} \alpha_0 & \text{first own of lag the dependent variable;} \\ 0 & \text{otherwise} \end{cases}$$

- $\underline{V}$, is diagonal with variance of $(i.l)$ element of A_j

$$V_{(i,l);j} = \begin{cases} \left(\frac{\pi_1}{j^{\pi_3}}\right)^2, & i = l \\ \left(\frac{\pi_1 \pi_2 \sigma_i}{j^{\pi_3} \sigma_l}\right)^2, & i \neq l \end{cases}$$

where π_1, π_2, π_3 are hyperparameters.

Example

Conjugate case

$$P(\alpha, \Sigma) = P(\alpha|\Sigma)P(\Sigma)$$
$$\alpha|\Sigma \sim N(\underline{\alpha}, \Sigma \otimes \underline{\underline{V}})$$
$$\Sigma \sim IW(\underline{S}, \underline{\nu})$$

The posteriors are

$$\alpha|\Sigma, y \sim N(\tilde{\alpha}, \Sigma \otimes \tilde{V})$$
$$\Sigma|y \sim IW(\tilde{S}, \tilde{\nu})$$

Details

Independent case

$$P(\alpha, \Sigma) = P(\alpha)P(\Sigma)$$
$$\alpha \sim N(\underline{\alpha}, \underline{V})$$
$$\Sigma \sim IW(\underline{S}, \underline{\nu})$$

In this case posteriors have no closed form, however **Gibbs sampling** can be effectively used.

$$\alpha|\Sigma, y \sim N(\tilde{\alpha}, \tilde{V})$$
$$\Sigma|\alpha, y \sim IW(\tilde{S}, \tilde{\nu})$$

Details

Posterior coefficients

- Fixed Σ case:
  1. sample from $\alpha|y \sim \mathrm{MVN}$.
- Conjugate:
  1. sample from $\Sigma|y \sim \mathrm{IW}$;
  2. sample from $\alpha|\Sigma, y \sim \mathrm{MVN}$.
- Independent (Gibbs sampling):
  1. sample from $\Sigma|\alpha, y \sim \mathrm{IW}$;
  2. sample from $\alpha|\Sigma, y \sim \mathrm{MVN}$.

Posterior prediction

Given the sampled parameters α, Σ, generate $\varepsilon_{T+1}, ..., \varepsilon_{T+H}$ from $\varepsilon_t \sim N(0, \Sigma)$ and compute recursively:

$$\tilde{y}_{T+h} = a_0 + \sum_{j=1}^{\min\{h-1;p\}} A_j \tilde{y}_{t+h-j} + \sum_{j=h}^{p} A_j y_{t+h-j} + \varepsilon_{T+h}$$

with $h = 1, \ldots, H$.

Posterior irfs

Given the sampled α, Σ, compute the VMA form: define recursively for each variable j in y_t

$$\tilde{y}^{(j)}_{t+h} = \sum_{l=1}^{\min\{h-1;p\}} A_l \tilde{y}_{t+h-l} + \sum_{l=h}^{p} A_l y_{t+h-l}$$

with $h = 1, \ldots, H$ and $y_{j,t} = 1; y_{i \neq j,t} = 0$.

Cholesky structuralization

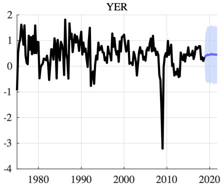
Recover from sampled $\Sigma = BB'$, then multiply the above irfs by B.

Replication of BEAR BVAR example

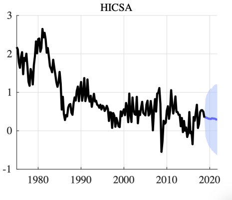
US macroeconomic quarterly data from 1974:q1 to 2018:q4

- Variables:
  1. `YER`, real GDP (growth);
  2. `HICSA`, inflation rate;
  3. `STN`, real interest rate.
- Lags $= 4$
- prior set-up : conjugate with Minnesota
  1. $\alpha_0 = 0.8$;
  2. $\pi_1 = 0.1$, $\pi_3 = 1$
  3. $\pi_4 = 100$
  4. AR residuals for Σ scale

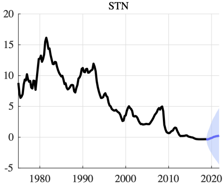
Forecast



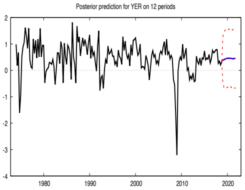
(a) MATLAB - YER



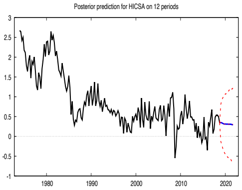
(b) MATLAB - HICSA



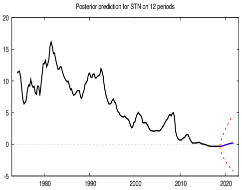
(c) MATLAB - STN



(d) BVAR - YER

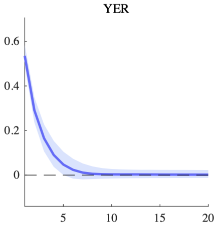


(e) BVAR - HICSA

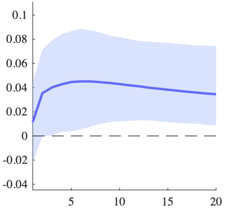


(f) BVAR - STN

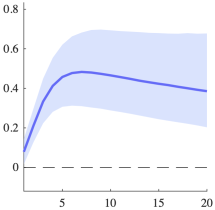
sirf - YER



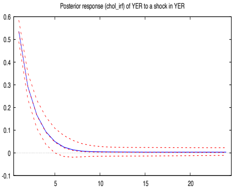
(a) YER→YER



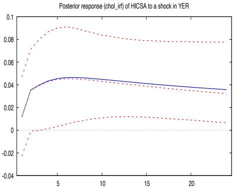
(b) YER→HICSA



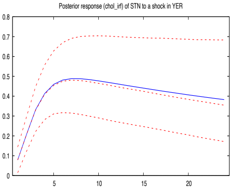
(c) YER→STN



(d) YER→YER

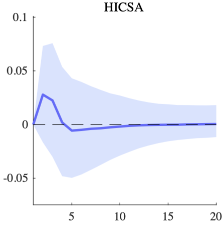


(e) YER→HICSA

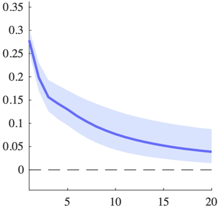


(f) YER→STN

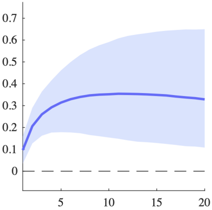
sirf - HICSA



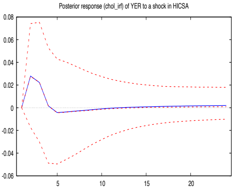
(a) HICSA→YER



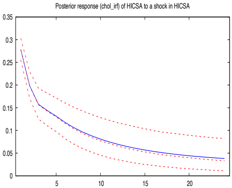
(b) HICSA→HICSA



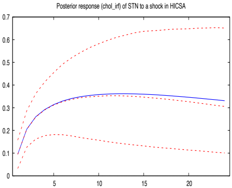
(c) HICSA→STN



(d) HICSA→YER

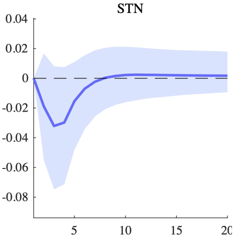


(e) HICSA→HICSA

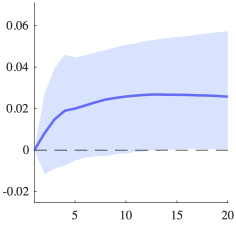


(f) HICSA→STN

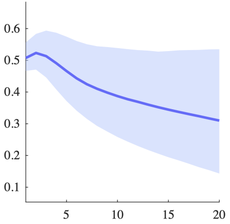
sirf - STN



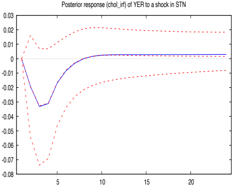
(a) STN→YER



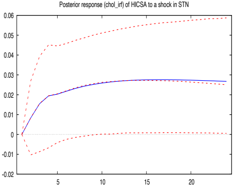
(b) STN→HICSA



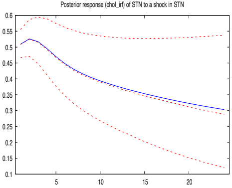
(c) STN→STN



(d) STN→YER



(e) STN→HICSA



(f) STN→STN

Conclusion

- `BVAR` introduces Bayesian VAR in `gretl`;
- different set-ups and computational tools are available;
- predictions and irfs are also available;
- successfully replicated BEAR basic usage.

What's next?

- `gridplot` integration;
- static forecast;
- more priors (hyperprior, shrinkage priors);
- long-run, sign and mixed restrictions;
- Historical decomposition and FEVD;
- ...

References

Bańbura, M., D. Giannone, and L. Reichlin (2010). Large bayesian vector auto regressions. *Journal of applied Econometrics 25*(1), 71–92.

Karlsson, S. (2013). Forecasting with bayesian vector autoregression. *Handbook of economic forecasting 2*, 791–897.

Koop, G. and D. Korobilis (2010). *Bayesian multivariate time series methods for empirical macroeconomics*. Now Publishers Inc.

Litterman, R. B. (1986). Forecasting with bayesian vector autoregressions—five years of experience. *Journal of Business & Economic Statistics 4*(1), 25–38.

Minnesota prior example

Consider a VAR(2) with $y_t = \{y_{1t}, y_{2t}\}$:

$$y_{1,t} = a_{1,0} + a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + a_{1,3}y_{1,t-2} + a_{1,4}y_{2,t-2} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,0} + a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + a_{2,3}y_{1,t-2} + a_{2,4}y_{2,t-2} + \varepsilon_{2,t}$$

Assume moreover

$$\alpha = [a_{1,0};\ a_{1,1};\ \ldots;\ a_{1,4};\ a_{2,0};\ a_{2,1};\ \ldots;\ a_{2,4}]'$$

$$\alpha \sim \mathrm{MVN}(\underline{\alpha}, \underline{V})$$

$$\underline{\alpha} = \begin{array}{cl} \begin{bmatrix} 0 \\ \alpha_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} (a_{1,0}) \\ (a_{1,1}) \\ (a_{1,2}) \\ (a_{1,3}) \\ (a_{1,4}) \\ (a_{2,0}) \\ (a_{2,1}) \\ (a_{2,2}) \\ (a_{2,3}) \\ (a_{2,4}) \end{matrix} \end{array}; \qquad diag(\underline{V}) = \begin{array}{cl} \begin{bmatrix} \sigma_1^2(\pi_1\pi_4)^2 \\ \pi_1^2 \\ \frac{\sigma_1^2}{\sigma_2^2}(\pi_1\pi_2)^2 \\ \left(\frac{\pi_1}{2^{\pi_3}}\right)^2 \\ \frac{\sigma_1^2}{\sigma_2^2}\left(\frac{\pi_1\pi_2}{2^{\pi_3}}\right)^2 \\ \sigma_1^2(\pi_1\pi_4)^2 \\ \frac{\sigma_2^2}{\sigma_1^2}(\pi_1\pi_2)^2 \\ \pi_1^2 \\ \frac{\sigma_2^2}{\sigma_1^2}\left(\frac{\pi_1\pi_2}{2^{\pi_3}}\right)^2 \\ \left(\frac{\pi_1}{2^{\pi_3}}\right)^2 \end{bmatrix} & \begin{matrix} (a_{1,0}) \\ (a_{1,1}) \\ (a_{1,2}) \\ (a_{1,3}) \\ (a_{1,4}) \\ (a_{2,0}) \\ (a_{2,1}) \\ (a_{2,2}) \\ (a_{2,3}) \\ (a_{2,4}) \end{matrix} \end{array}$$

Conjugate posteriors

$$\alpha|\Sigma, y \sim N(\tilde{\alpha}, \Sigma \otimes \tilde{V})$$
$$\Sigma|y \sim IW(\tilde{S}, \tilde{\nu})$$

where

$$\tilde{V} = [\underline{\underline{V}}^{-1} + X'X]^{-1},$$
$$\tilde{\alpha} = vec(\tilde{A}) \rightarrow \tilde{A} = \tilde{V}[\underline{\underline{V}}^{-1}\underline{A} + X'X\hat{A}]$$
$$\tilde{S} = \hat{S} + \underline{S} + \hat{A}'X'X\hat{A} + \underline{A}'\underline{\underline{V}}^{-1}\underline{A} - \tilde{A}(\underline{\underline{V}}^{-1} + X'X)\tilde{A}$$
$$\tilde{\nu} = T + \underline{\nu}$$

Back

Independent conditional posteriors

$$\alpha|\Sigma, y \sim N(\tilde{\alpha}, \tilde{V})$$
$$\Sigma|\alpha, y \sim IW(\tilde{S}, \tilde{\nu})$$

where:

$$\tilde{V} = \left[\underline{V}^{-1} + \Sigma^{-1} \otimes X'X\right]^{-1},$$
$$\tilde{\alpha} = \tilde{V}\left[\underline{V}^{-1}\underline{\alpha} + (\Sigma^{-1} \otimes X')y\right],$$
$$\tilde{S} = \underline{S} + (Y - X\tilde{A})'(Y - X\tilde{A}),$$
$$\tilde{\nu} = T + \underline{\nu}$$

with $\tilde{A} = unvec(\tilde{\alpha})$.

Back