DIW GRADUATE CENTER

Structural Vector Autoregressions with Heteroskedasticity

A comparison of Different Volatility Models

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Outline

VAR model

Variables of interest $y_t = (y_{1t}, \dots, y_{Kt})'$

Reduced form VAR

$$y_t = \nu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t$$

- $u_t \sim (0, \Sigma_u)$

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Structural errors

- $\varepsilon_t = \mathrm{B}^{-1} u_t$
- B – matrix of impact effects of shocks
- $\varepsilon_t \sim (0, \Sigma_\varepsilon)$
- Σ_ε diagonal $\quad (\Sigma_\varepsilon = I_K)$
- $\Sigma_u = \mathrm{B}\Sigma_\varepsilon \mathrm{B}' \quad (\Sigma_u = \mathrm{BB}')$

Identification of Shocks via Change in Volatility

A matrix decomposition result

Σ_1, Σ_2 positive definite

$\Rightarrow \exists$ a $(K \times K)$ matrix B and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_K)$

such that $\Sigma_1 = BB'$ and $\Sigma_2 = B\Lambda B'$

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B is (locally) unique if the λ_j's are distinct

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Identification assumption

Instantaneous effects of shocks are the same across all states

More volatility states

Finite number of states

$\Sigma_1 = BB'$, $\Sigma_m = B\Lambda_m B'$, $m = 2, \dots, M$,
$\Lambda_m = \text{diag}(\lambda_{m1}, \dots, \lambda_{mK})$ $(m = 2, \dots, M)$ diagonal matrices.

Representation may not exist if arbitrary covariance matrices Σ_m $(m = 1, \dots, M)$ are allowed for.

Uniqueness of B (apart from ordering and sign) if for any two subscripts $k, l \in \{1, \dots, K\}$, $k \neq l$, there is a $j \in \{2, \dots, M\}$ such that $\lambda_{jk} \neq \lambda_{jl}$.

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Continuous number of states

$\Sigma_t = B\Lambda_t B', \quad \Lambda_t$ diagonal.

Exogenous changes in volatility

General setup

(Rigobon (2003), Rigobon and Sack (2003), Lanne and Lütkepohl (2008))

$$\mathbb{E}(u_t u_t') = \Sigma_t = \begin{cases} \Sigma_1 & \text{for} \quad t \in \mathrm{T}_1, \\ \vdots & \\ \Sigma_M & \text{for} \quad t \in \mathrm{T}_M, \end{cases}$$

$\mathrm{T}_m = \{T_{m-1}+1, \dots, T_m\}$ $(m = 1, \dots, M)$ are M given volatility regimes.

$T_0 = 0$ and $T_M = T$.

The T_m represent the points of volatility changes.

Estimation

ML estimation

$$\log l(\beta, \boldsymbol{\sigma}) =$$
$$-\frac{KT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log|\Sigma_t| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Sigma_t^{-1}u_t$$

$\boldsymbol{\sigma}-$ covariance parameters

$\beta = \text{vec}[\nu, A_1, \dots, A_p]$

GLS estimation

$\hat{\Sigma}_m = \frac{1}{T_m - T_{m-1}}\sum_{t\in \mathrm{T}_m}\hat{u}_t\hat{u}_t'$ ($\hat{u}_t$ OLS residuals)

$$\hat{\beta} =$$
$$\left(\sum_{t=1}^{T} Z_{t-1}Z_{t-1}' \otimes \hat{\Sigma}_t^{-1}\right)^{-1}\left(\sum_{t=1}^{T}(Z_{t-1}\otimes\hat{\Sigma}_t^{-1})y_t\right)$$

$\hat{\Sigma}_t = \hat{\Sigma}_m$ for $t \in \mathrm{T}_m$

$Z_{t-1} = (1, y_{t-1}', \dots, y_{t-p}')'$

Identification via Heteroskedasticity

Null hypotheses

$$H_0 : \lambda_{mi} = \lambda_{mj}, \quad m = 2, \ldots, M,$$

have to be rejected $\forall i, j, i \neq j$.

Tests

LR tests with asymptotic $\chi^2(2)$ distributions (properties of tests under investigation).

Markov switching in residual volatility (Lanne, Lütkepohl, Maciejowska 2010; Herwartz, Lütkepohl 2014)

Markov process

$$s_t \ (t = 0, \pm 1, \pm 2, \dots)$$
$$s_t \in \{1, \dots, M\}$$

Transition probabilities

$$p_{ij} = \Pr(s_t = j | s_{t-1} = i),$$
$$i, j = 1, \dots, M$$

Reduced form residuals

$$u_t | s_t \sim \mathcal{N}(0, \Sigma_{s_t})$$

ML estimation

Log likelihood function

$$\log l(\beta, \mathrm{B}, \boldsymbol{\lambda}, P|\mathbf{y}) = \sum_{t=1}^{T} \log \left(\sum_{m=1}^{M} \Pr(s_t = m | Y_{t-1}) f(y_t | s_t = m, Y_{t-1}) \right)$$

$\boldsymbol{\lambda}$ – vector of all diagonal elements of $\Lambda_2, \dots, \Lambda_M$
P – matrix of transition probabilities
$\mathbf{y}$ – full sample
$Y_{t-1} = (y'_{t-1}, \dots, y'_{t-p})'$

$$f(y_t | s_t = m, Y_{t-1}) = (2\pi)^{-K/2} \det(\Sigma_m)^{-1/2} \exp\left\{ -\frac{1}{2} u'_t \Sigma_m^{-1} u_t \right\}$$

ML estimation

Related problems

- Ordering of λ_{ij}'s
- Sign of shocks
- Label switching
- Many local optima of log likelihood
- Variances have to be bounded away from zero
- Covariance matrices have to be bounded away from singularity

Inference

Identification

LR tests as in case of exogenous volatility changes.

Estimation of impulse responses

(Herwartz and Lütkepohl (2014))

Fixed design wild bootstrap
$y_t^* = \hat{\nu} + \hat{A}_1 y_{t-1} + \cdots + \hat{A}_p y_{t-p} + u_t^*$

$u_t^* = \eta_t \hat{u}_t$, where η_t is a binary random variable with values 1 and -1 that have equal probability

Bootstrap parameter estimates θ^* of $\theta = \text{vec}[\nu, A_1, \dots, A_p]$ and B* of B, conditionally on the initially estimated transition probabilities and $\boldsymbol{\lambda}$

Smooth transition in residual covariances

Reduced form residual covariance

(Lütkepohl and Netšunajev (2014))

$$\mathbb{E}(u_t u_t') = \Omega_t = (1 - G(\gamma, c, s_t))\Sigma_1 + G(\gamma, c, s_t)\Sigma_2$$
$$= (1 - G(\gamma, c, s_t))BB' + G(\gamma, c, s_t)B\Lambda B'$$

Transition function

$s_t = t$ or some (lagged) economic variable

$$G(\gamma, c, s_t) = (1 + \exp[-\exp(\gamma)(s_t - c)])^{-1}$$

$$0 < G(\gamma, c, s_t) < 1$$

ML Estimation

Likelihood function

$$\log l = \text{constant} - \frac{1}{2}\sum_{t=1}^{T}\log\det(\Omega_t) - \frac{1}{2}\sum_{t=1}^{T} u_t'\Omega_t^{-1}u_t$$

Optimization Step 1:

Given $\nu, A_1, \dots, A_p$, γ, and c, maximize over B, Λ possibly subject to restrictions

Optimization Step 2:

Given γ, c and B, Λ the model is linear. Use grid over reasonable range for c and γ.

Identification testing

Based on decomposition $\Sigma_1 = \mathrm{BB}'$, $\Sigma_2 = \mathrm{B}\Lambda\mathrm{B}$.
LR tests for equality of diagonal elements of Λ.

Bootstrapping Impulse Responses

Fixed design wild bootstrap

$$y_t^* = \hat{\nu} + \hat{A}_1 y_{t-1} + \cdots + \hat{A}_p y_{t-p} + u_t^*$$

- $u_t^* = \eta_t \hat{u}_t$
- η_t has Rademacher distribution

Conditioning

Condition on initially estimated transition parameters γ, c

Vector GARCH residuals

General setup (Normandin and Phaneuf (2004) etc.)

$$\Sigma_{u,t|t-1} = \mathbb{E}(u_t u_t' | u_{t-1}, \dots) = \mathrm{B}\Sigma_{\varepsilon,t|t-1}\mathrm{B}'$$

$\Sigma_{\varepsilon,t|t-1} = \mathrm{diag}(\sigma^2_{1,t|t-1}, \dots, \sigma^2_{K,t|t-1})$ is a diagonal matrix with
$\sigma^2_{k,t|t-1} = (1 - \gamma_k - g_k) + \gamma_k \varepsilon^2_{t-1} + g_k \sigma^2_{k,t-1|t-2}$,
$k = 1, \dots, K$.

Identification (Sentana and Fiorentini (2001))

Γ such that $(\sigma^2_{k,1|0}, \dots, \sigma^2_{k,T|T-1})$ is the kth row of Γ'

$\Gamma'\Gamma$ invertible

$\Rightarrow$ identification of B (apart from changes in sign and permutation of columns).

Vector GARCH residuals (cont'd)

Alternative setup

$$u_t = \mathrm{B} \begin{bmatrix} \Lambda_{t|t-1}^{1/2} & 0 \\ 0 & I_{K-r} \end{bmatrix} e_t,$$

$e_t \sim \mathrm{iid}(0, I_K)$ structural errors

$$\Lambda_{t|t-1} = \begin{bmatrix} \sigma^2_{1,t|t-1} & & 0 \\ & \ddots & \\ 0 & & \sigma^2_{r,t|t-1} \end{bmatrix}$$

$$u_{t|t-1} \sim \left(0, \Sigma_{t|t-1} = \mathrm{B} \begin{bmatrix} \Lambda_{t|t-1} & 0 \\ 0 & I_{K-r} \end{bmatrix} \mathrm{B}'\right)$$

Identification B unique up to column sign and permutation if $r \geq K - 1$ (Milunovich, Yang 2013)

Further Specification Details

Setup details

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} = \mathrm{B}^{-1}$$

$$\mathbf{A}_1 u_t = \Lambda_{t|t-1}^{1/2} e_{1t} \quad \text{and} \quad \mathbf{A}_2 u_t = e_{2t}$$

GARCH specification

$$\sigma_{k,t|t-1}^2 = (1 - \gamma_k - g_k) + \gamma_k (\mathbf{a}_k u_{t-1})^2 + g_k \sigma_{k,t-1|t-2}^2$$

$\mathbf{a}_k$ is kth row of $\mathbf{A}_1$, $\gamma_k \neq 0$

Unconditional covariance $\Sigma_u = \mathrm{BB}'$

Polar decomposition of B

$\mathrm{B} = CR$

C symmetric, positive definite

$R = [R_1 : R_2]$ orthogonal

Conditional covariance $\Sigma_{t|t-1} = \Sigma_u + CR_1(\Lambda_{t|t-1} - I_r)R_1' C$

Likelihood Function (Lanne, Saikkonen 2007)

$$\log l = \sum_{t=1}^{T} \log f_{t|t-1}(y_t)$$

$$\begin{aligned} f_{t|t-1}(y_t) &= (2\pi)^{-K/2} \det(\Sigma_{t|t-1})^{-1/2} \exp\left(-\frac{1}{2} u_t' \Sigma_{t|t-1}^{-1} u_t\right) \\ &= (2\pi)^{-K/2} \det(\Sigma_u)^{-1/2} \exp\left(-\frac{1}{2} u_t' \Sigma_u^{-1} u_t\right) \prod_{k=1}^{r} \sigma_{k,t|t-1}^{-1} \\ &\quad \times \exp\left(-\frac{1}{2} u_t' C^{-1} R_1 (\Lambda_{t|t-1}^{-1} - I_r) R_1' C^{-1} u_t\right). \end{aligned}$$

Depends on $\nu, A_1, \ldots, A_p$, C, R_1, and the GARCH parameters only, and not on unidentified R_2.

Identification Tests (Lanne, Saikkonen 2007; Lütkepohl, Milunovich 2015)

$\tilde{R}_2 = \tilde{R}_{1\perp}(\tilde{R}'_{1\perp}\tilde{R}_{1\perp})^{-1/2}, \quad \tilde{\mathbf{A}}_2 = \tilde{R}'_2\tilde{C}^{-1}u_t$
estimates linear transformation of e_{2t}.
Base tests on autorcorrelations of squared components.

Studies on Monetary Policy and Stock Market

Short-run restrictions

Li, Iscan, Xu (2010, JIMF); Thorbecke (1997, JoFinance); Park, Ratti (2000, FR); Patelis (1997, JoFinance)

Long-run restrictions

Lastrapes (1998, JIMF); Rapach (2001, JEB)

Mixed short-run and long-run restrictions

Bjørnland, Leitemo (2009, JME)

Results

Monetary shocks affect stock prices, the magnitude varies across studies

Bjørnland-Leitemo Model

Variables used

$$y_t = (q_t, \pi_t, c_t, \Delta sp_t, r_t)'$$

- q_t industrial production index;
- π_t CPI inflation ($\times 100$);
- c_t commodity price index ($\times 100$);
- sp_t log real S&P500 stock price index used as monthly returns (Δsp_t);
- r_t Federal Funds rate.

Data

Monthly data for 1970M1 - 2007M6

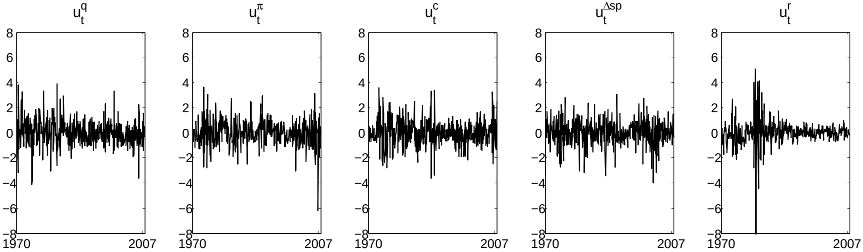
Conventional Identification (Bjørnland-Leitemo)

$$\begin{bmatrix} q_t \\ \pi_t \\ c_t \\ \Delta sp_t \\ r_t \end{bmatrix} \mathrm{B} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \Xi_\infty = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \\ \varepsilon_t^{sp} \\ \varepsilon_t^{m} \end{bmatrix}$$

Alternative Sets of Restrictions

- M1: B lower triangular (recursive identification);
- M2: B and Ξ_∞ restricted as above (Bjørnland-Leitemo identification);
- M3: only the two last columns of B and Ξ_∞ restricted as in (Bjørnland-Leitemo);
- M4: only B restricted as in (Bjørnland-Leitemo).

Residuals obtained from the VAR(3) model

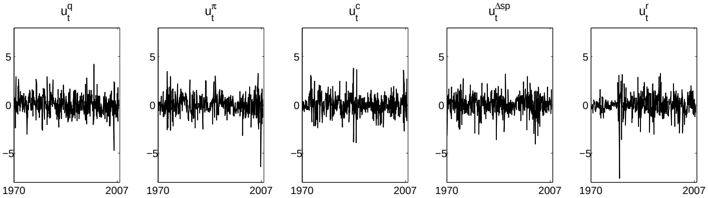


Comparison of SVAR(3) Models with State Invariant B

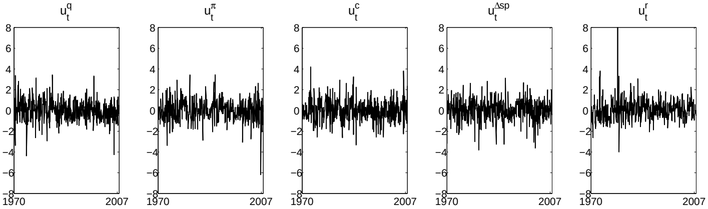
Model	$\log L_T$	AIC	SC
SVAR(3)	-3159.344	6508.689	6899.067
ST-SVAR(3)	-2878.255	5976.510	6428.527
MS(2)-SVAR(3)	-2826.742	5877.484	6337.719
MS(3)-SVAR(3)	-2774.614	5791.230	6288.448
SVAR(3)-GARCH(1,1)	-2891.971	6013.942	6486.505

Note: L_T – likelihood function, AIC $= -2\log L_T + 2\times$no of free parameters,
SC $= -2\log L_T + \log T\times$no of free parameters.

Standardized residuals obtained from different models

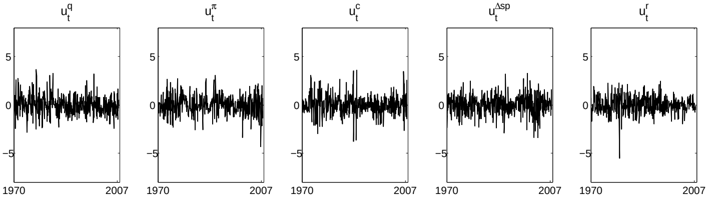


(a) Standardized residuals obtained from ST-SVAR model

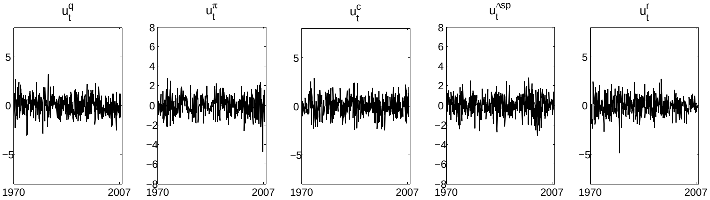


(b) Standardized residuals obtained from SVAR-GARCH model

Standardized residuals obtained from different models



(c) Standardized residuals obtained from MS(2)-SVAR model



(d) Standardized residuals obtained from MS(3)-SVAR model

Estimates of Relative Variances of ST-SVAR and MS-SVAR Models

	ST-SVAR		MS(2)-SVAR		MS(3)-SVAR	
parameter	estimate	std.dev.	estimate	std.dev.	estimate	std.dev.
λ_{21}	0.019	0.002	0.019	0.003	3.724	0.849
λ_{22}	0.315	0.057	0.271	0.097	2.741	0.626
λ_{23}	0.548	0.088	0.371	0.125	5.243	1.309
λ_{24}	0.867	0.154	0.428	0.135	3.544	0.761
λ_{25}	0.927	0.172	0.682	0.356	2.568	0.608
λ_{31}					1.859	0.593
λ_{32}					2.554	0.747
λ_{33}					2.693	0.863
λ_{34}					3.981	1.153
λ_{35}					82.265	16.065

Tests for Identification in GARCH-SVAR Model

r under H_0	$Q_1(1)$	df	p-value	$Q_2(1)$	df	p-value
1	20.026	1	9.5×10^{-7}	251.520	100	4.8×10^{-15}
2	20.446	1	6.1×10^{-6}	148.644	36	1.2×10^{-15}
3	12.712	1	3.6×10^{-4}	31.256	9	2.6×10^{-4}
4	15.127	1	1.1×10^{-4}	15.127	1	1.1×10^{-4}

Tests for Conventional Identifying Restrictions in Heteroskedastic SVAR Models

H_0	H_1	LR stat	df	p-value
M1	ST-SVAR with unrestricted B	23.395	10	0.009
M2	ST-SVAR with unrestricted B, Ξ_∞	35.845	10	8.9×10^{-5}
M3	ST-SVAR with unrestricted B, Ξ_∞	30.909	7	6.4×10^{-5}
M4	ST-SVAR with unrestricted B, Ξ_∞	22.491	9	0.0074
M2	M4	13.354	1	2.5×10^{-4}
M1	SVAR-GARCH with unrestricted B	55.069	10	2.8×10^{-8}
M2	SVAR-GARCH with unrestricted B, Ξ_∞	90.047	10	3.7×10^{-15}
M3	SVAR-GARCH with unrestricted B, Ξ_∞	92.229	7	0.000
M4	SVAR-GARCH with unrestricted B, Ξ_∞	56.052	9	6.7×10^{-9}
M2	M4	33.995	1	4.5×10^{-9}

Tests for Conventional Identifying Restrictions in Heteroskedastic SVAR Models

H_0	H_1	LR stat	df	p-value
M1	MS(2)-SVAR with unrestricted B	19.101	10	0.039
M2	MS(2)-SVAR with unrestricted B, Ξ_∞	52.505	10	9.1×10^{-8}
M3	MS(2)-SVAR with unrestricted B, Ξ_∞	54.586	7	1.4×10^{-13}
M4	MS(2)-SVAR with unrestricted B, Ξ_∞	19.097	9	0.024
M2	M4	33.408	1	7.4×10^{-9}
M1	MS(3)-SVAR with unrestricted B	28.039	10	0.0018
M2	MS(3)-SVAR with unrestricted B, Ξ_∞	54.134	10	4.5×10^{-8}
M3	MS(3)-SVAR with unrestricted B, Ξ_∞	49.350	7	1.93×10^{-8}
M4	MS(3)-SVAR with unrestricted B, Ξ_∞	27.921	9	9.8×10^{-4}
M2	M4	26.212	1	3.5×10^{-7}

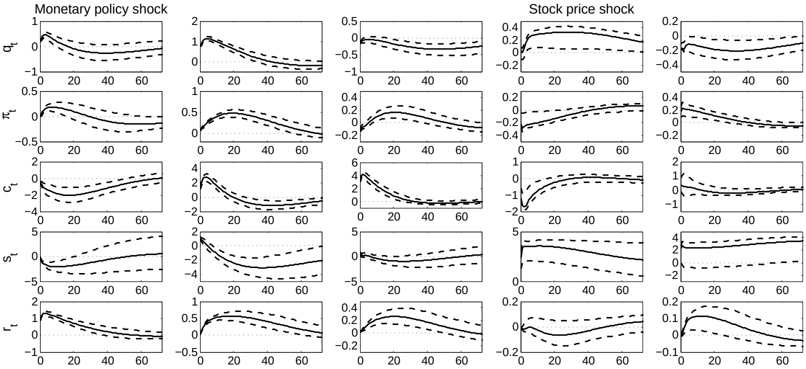


Figure: Impulse response functions for fully unrestricted ST-SVAR model. Solid line - point estimate of the response, dashed line - 68% confidence bands based on 1000 bootstrap replications.

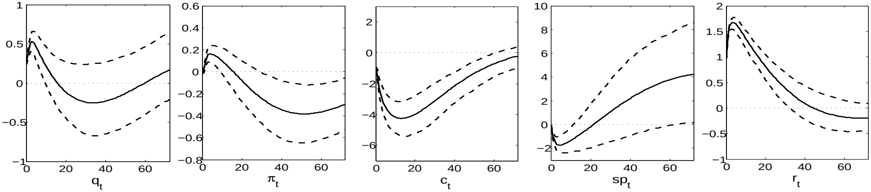


Figure: Impulse Responses to Monetary Shock Obtained from MS(3)-SVAR model

Research Problems

- Estimation of larger models with (conditional) heteroskedasticity
- Bayesian estimation
- Identification tests
- Specification of suitable models
- Confidence bands for impulse responses
- Other volatility models