# Reconciling TEV and VaR in Active Portfolio Management: A New Frontier

Jack Lucchetti, Mihaela Nicolau, Giulio Palomba, Luca Riccetti

7th online Gretl Conference, 3rd June 2021

# Issued at the Gretl Conference: it's a must!

Historically, our resarch topics were presented for the 1st time in the framework of the `Gretl` Conference

2011 - Torún, Poland Palomba & Riccetti (Journal of Banking and Finance, 2012),

2013 - Oklahoma City, USA Palomba & Riccetti (Journal of Risk and Financial Management, 2019),

2021 - my house Lucchetti, Nicolau, Palomba & Riccetti (preliminary draft),

# Highlights

- we investigate the risk-return relationship of active portfolios,

# Highlights

- we investigate the risk-return relationship of active portfolios,
- we focus on portfolio frontiers in presence of restrictions on the Tracking Error Volatility (**TEV**)

$$\mathrm{TEV} = (\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)'\boldsymbol{\Sigma}(\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)$$

and on the Value-at-Risk (**VaR**)

$$\mathrm{VaR} = z_\theta \sigma_P - \mu_P,$$

where

- $\boldsymbol{\omega}_P$ and $\boldsymbol{\omega}_B$ are $n \times 1$ vectors containing the portfolio weights,
- $P$ is the general portfolio and $B$ is the benchmark,
- $\boldsymbol{\Sigma}$ is the covariance matrix of the $n$ risky assets,
- $z_\theta$ is the standard normal quantile (with $0.5 \leq \theta < 1$),
- $\sigma_P$ and $\mu_P$ are the risk and the expected return of portfolio $P$.

# Highlights

- we investigate the risk-return relationship of active portfolios,
- we focus on portfolio frontiers in presence of restrictions on the Tracking Error Volatility (**TEV**)

$$\mathrm{TEV} = (\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)'\boldsymbol{\Sigma}(\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)$$

  and on the Value-at-Risk (**VaR**)

$$\mathrm{VaR} = z_\theta \sigma_P - \mu_P,$$

  where
  - $\boldsymbol{\omega}_P$ and $\boldsymbol{\omega}_B$ are $n \times 1$ vectors containing the portfolio weights,
  - $P$ is the general portfolio and $B$ is the benchmark,
  - $\boldsymbol{\Sigma}$ is the covariance matrix of the $n$ risky assets,
  - $z_\theta$ is the standard normal quantile (with $0.5 \leq \theta < 1$),
  - $\sigma_P$ and $\mu_P$ are the risk and the expected return of portfolio $P$.
- we focus on the trade-off between the above mentioned restrictions,

# Highlights

- we investigate the risk-return relationship of active portfolios,
- we focus on portfolio frontiers in presence of restrictions on the Tracking Error Volatility (**TEV**)

$$\text{TEV} = (\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)'\boldsymbol{\Sigma}(\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)$$

and on the Value-at-Risk (**VaR**)

$$\text{VaR} = z_\theta \sigma_P - \mu_P,$$

where

- $\boldsymbol{\omega}_P$ and $\boldsymbol{\omega}_B$ are $n \times 1$ vectors containing the portfolio weights,
- $P$ is the general portfolio and $B$ is the benchmark,
- $\boldsymbol{\Sigma}$ is the covariance matrix of the $n$ risky assets,
- $z_\theta$ is the standard normal quantile (with $0.5 \leq \theta < 1$),
- $\sigma_P$ and $\mu_P$ are the risk and the expected return of portfolio $P$.

- we focus on the trade-off between the above mentioned restrictions,
- we define a new portfolio frontier,

# Highlights

- we investigate the risk-return relationship of active portfolios,
- we focus on portfolio frontiers in presence of restrictions on the Tracking Error Volatility (**TEV**)

$$\mathrm{TEV} = (\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)' \boldsymbol{\Sigma} (\boldsymbol{\omega}_P - \boldsymbol{\omega}_B)$$

  and on the Value-at-Risk (**VaR**)

$$\mathrm{VaR} = z_\theta \sigma_P - \mu_P,$$

  where
  - $\boldsymbol{\omega}_P$ and $\boldsymbol{\omega}_B$ are $n \times 1$ vectors containing the portfolio weights,
  - $P$ is the general portfolio and $B$ is the benchmark,
  - $\boldsymbol{\Sigma}$ is the covariance matrix of the $n$ risky assets,
  - $z_\theta$ is the standard normal quantile (with $0.5 \leq \theta < 1$),
  - $\sigma_P$ and $\mu_P$ are the risk and the expected return of portfolio $P$.
- we focus on the trade-off between the above mentioned restrictions,
- we define a new portfolio frontier,
- we provide economic/financial implications.

# Basic assumptions

- geometric analysis in the usual $(\sigma_P, \mu_P)$ space,
- $n$ risky assets,
- short sales allowed,
- quadratic utility function,
- normally distributed returns.

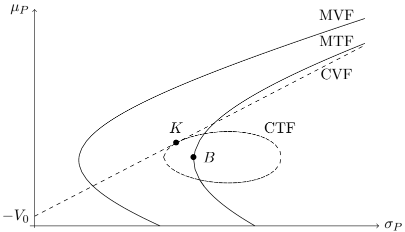
Portfolio frontiers:

- Mean-Variance Frontier (**MVF**, Markowitz, 1959),
- Mean-TEV Frontier (**MTF**, Roll, 1992)
- Constrained TEV Frontier (**CTF**, Jorion, 2003),
- Constrained VaR Frontier (**CVF**, Alexander & Batista, 2008).

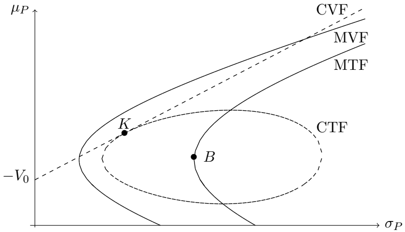
# Tangency portfolio $K$

Aim of the paper:
identify a subset of efficient VaR-TEV portfolios.

(a) stringent TEV,
weak VaR restriction




(b) stringent VaR
weak TEV restriction




**TRADE OFF: we can not reduce the VaR without increasing the TEV and vice versa.**

# Derivation of the RBF

$$\begin{array}{ll} \min & \mathrm{VaR} = z_\theta \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} - \boldsymbol{\omega}'\boldsymbol{\mu} \\ \mathrm{sub} & \sqrt{(\boldsymbol{\omega} - \boldsymbol{\omega}_B)'\boldsymbol{\Sigma}(\boldsymbol{\omega} - \boldsymbol{\omega}_B)} = \sqrt{T_0} \\ & \boldsymbol{\omega}'\boldsymbol{\iota} = 1, \end{array}$$

where

- $T_0$ is the TEV restriction,
- $\boldsymbol{\mu}$ is the $n$-dimensional vector containing the risky asset returns,
- $\sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} = \sigma_P$ and $\boldsymbol{\omega}'\boldsymbol{\mu} = \mu_P$,
- $\boldsymbol{\iota} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}'$ ha dimension $n \times 1$.

# Derivation of the RBF

$$\begin{array}{ll} \min & \mathrm{VaR} = z_\theta \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} - \boldsymbol{\omega}'\boldsymbol{\mu} \\ \mathrm{sub} & \sqrt{(\boldsymbol{\omega}-\boldsymbol{\omega}_B)'\boldsymbol{\Sigma}(\boldsymbol{\omega}-\boldsymbol{\omega}_B)} = \sqrt{T_0} \\ & \boldsymbol{\omega}'\boldsymbol{\iota} = 1, \end{array}$$

where

- $T_0$ is the TEV restriction,
- $\boldsymbol{\mu}$ is the $n$-dimensional vector containing the risky asset returns,
- $\sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} = \sigma_P$ and $\boldsymbol{\omega}'\boldsymbol{\mu} = \mu_P$,
- $\boldsymbol{\iota} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}'$ ha dimension $n \times 1$.

Solutions (after some algebra):

$$\begin{cases} \boldsymbol{\omega}^* = x_1(\boldsymbol{\omega})\,\boldsymbol{\omega}_C + x_2(\boldsymbol{\omega})\,\boldsymbol{\omega}_Q + x_3(\boldsymbol{\omega})\,\boldsymbol{\omega}_B & \text{optimal portfolio weights} \\ \lambda_1^* = \dfrac{\sqrt{T_0}}{\mu_P - \mu_B}\left[\dfrac{z_\theta}{a\sigma_P}(\mu_P - \mu_C) - d\right] & \dfrac{\partial\,\mathrm{VaR}}{\partial T_0} \text{ (1st Lagrange m.)} \\ \lambda_2^* = \dfrac{z_\theta}{a\sigma_P} - \mu_C & \text{(2nd Lagrange multiplier)} \end{cases}$$

where

- $\mu_B$ and $\mu_C$ are the benchmark and the GMV portfolio return,
- $a = \boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$, $b = \boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $c = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $d = c - b^2/a$

# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{Wx}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\omega_C$ - Global Minimum Variance,
   - $\omega_Q$ - Optimal Sharpe Ratio,
   - $\omega_B$ - Benchmark;

# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{W}\boldsymbol{x}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\boldsymbol{\omega}_C$ - Global Minimum Variance,
   - $\boldsymbol{\omega}_Q$ - Optimal Sharpe Ratio,
   - $\boldsymbol{\omega}_B$ - Benchmark;
2. $\boldsymbol{x}(\omega) \in \mathbb{R}^3$ and $\displaystyle\sum_{i=1}^{3} x_i(\omega) = 1$;

# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{W}\boldsymbol{x}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\omega_C$ - Global Minimum Variance,
   - $\omega_Q$ - Optimal Sharpe Ratio,
   - $\omega_B$ - Benchmark;
2. $\boldsymbol{x}(\omega) \in \mathbb{R}^3$ and $\sum_{i=1}^{3} x_i(\omega) = 1$;
3. benchmark portfolio: $\boldsymbol{x}(\omega) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;

# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{Wx}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\omega_C$ - Global Minimum Variance,
   - $\omega_Q$ - Optimal Sharpe Ratio,
   - $\omega_B$ - Benchmark;
2. $\boldsymbol{x}(\omega) \in \mathbb{R}^3$ and $\sum_{i=1}^{3} x_i(\omega) = 1$;
3. benchmark portfolio: $\boldsymbol{x}(\omega) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;
4. conversely, $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for portfolios $\omega_C$ and $\omega_Q$ (they belong to the MVF);

# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{Wx}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\omega_C$ - Global Minimum Variance,
   - $\omega_Q$ - Optimal Sharpe Ratio,
   - $\omega_B$ - Benchmark;
2. $\boldsymbol{x}(\omega) \in \mathbb{R}^3$ and $\sum_{i=1}^{3} x_i(\omega) = 1$;
3. benchmark portfolio: $\boldsymbol{x}(\omega) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;
4. conversely, $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for portfolios $\omega_C$ and $\omega_Q$ (they belong to the MVF);
5. $\omega^*$ does not depend on the restriction $\mathrm{TEV} = T_0$;

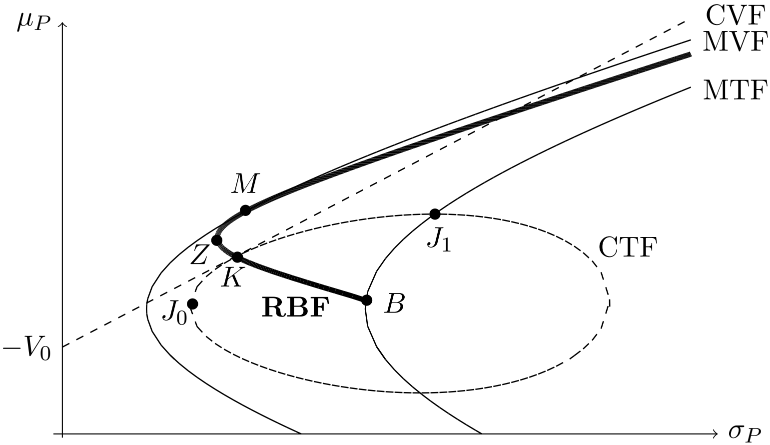
# Remarks

Starting from eq. $\omega^* = x_1(\omega)\,\omega_C + x_2(\omega)\,\omega_Q + x_3(\omega)\,\omega_B = \boldsymbol{Wx}(\omega)$ we prove that:

1. the portfolio $\omega^*$ is a linear combination of 3 portfolios:
   - $\omega_C$ - Global Minimum Variance,
   - $\omega_Q$ - Optimal Sharpe Ratio,
   - $\omega_B$ - Benchmark;
2. $\boldsymbol{x}(\omega) \in \mathbb{R}^3$ and $\sum_{i=1}^{3} x_i(\omega) = 1$;
3. benchmark portfolio: $\boldsymbol{x}(\omega) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;
4. conversely, $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}(\omega) \neq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for portfolios $\omega_C$ and $\omega_Q$ (they belong to the MVF);
5. $\omega^*$ does not depend on the restriction $\mathrm{TEV} = T_0$;
6. no closed-form definition for the RBF, it must be determined via numerical techniques.

# The Risk Balancing Frontier (RBF)

A new portfolio boundary in the standard $(\sigma_P, \mu_P)$ space



# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;

# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the “oval” CTF and the linear CVF;

# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the “oval” CTF and the linear CVF;
3. it “starts” from the benchmark (minimum $\mathrm{TEV} = 0$);

# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the "oval" CTF and the linear CVF;
3. it "starts" from the benchmark (minimum $\mathrm{TEV} = 0$);
4. the tangency point to the MVF is portfolio $M$ (minimum $\mathrm{VaR} = V_M$);

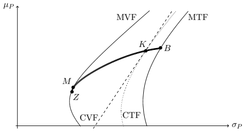
# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the "oval" CTF and the linear CVF;
3. it "starts" from the benchmark (minimum $\mathrm{TEV} = 0$);
4. the tangency point to the MVF is portfolio $M$ (minimum $\mathrm{VaR} = V_M$);
5. the portfolio $Z$ is the less risky (minimum variance on the RBF);

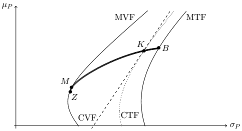
# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the “oval” CTF and the linear CVF;
3. it “starts” from the benchmark (minimum $\mathrm{TEV} = 0$);
4. the tangency point to the MVF is portfolio $M$ (minimum $\mathrm{VaR} = V_M$);
5. the portfolio $Z$ is the less risky (minimum variance on the RBF);
6. $\mu_P = g(T_0)$ is monotonic;

# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the "oval" CTF and the linear CVF;
3. it "starts" from the benchmark (minimum $\mathrm{TEV} = 0$);
4. the tangency point to the MVF is portfolio $M$ (minimum $\mathrm{VaR} = V_M$);
5. the portfolio $Z$ is the less risky (minimum variance on the RBF);
6. $\mu_P = g(T_0)$ is monotonic;
7. aggressive benchmarks ($\mu_B > \mu_M$)  ;

# Properties of the RBF

1. it contains all the portfolios characterised by the minimum VaR available for each TEV level;
2. it is an evelope of all tangency portfolios between the "oval" CTF and the linear CVF;
3. it "starts" from the benchmark (minimum $\mathrm{TEV} = 0$);
4. the tangency point to the MVF is portfolio $M$ (minimum $\mathrm{VaR} = V_M$);
5. the portfolio $Z$ is the less risky (minimum variance on the RBF);
6. $\mu_P = g(T_0)$ is monotonic;
7. aggressive benchmarks ($\mu_B > \mu_M$) 

;
8. the shape does not depend on
   - manager confidence level (slope of the CVF),
   - slope of the horizontal axis of the CTF.

# The algorithm

$$\begin{cases} \text{for} & T_0 = 0, h, 2h, 3h, \ldots, T^{\max} \\ \min\limits_{\mu_i} & \mathrm{VaR}_i = z_\theta S^{1/2}\left(T_0, \mu_i(T_0)\right) - \mu_i(T_0), \end{cases}$$

where

- $S^{1/2}(\cdot)$ is the portfolio risk over the arc $\widehat{J_0 J_1}$,
- $h$ : arbitrary and numerically small increment.

The algorithm proceeds via the following steps:

1. start from $\mathrm{TEV} = T_0$,
2. calculate the midpoint $0.5[\mu_{J0}(T_0) + \mu_{J1}(T_0)]$,
3. find the minimum VaR portfolio on $\widehat{J_0 J_1}$ via BFGS,
4. determine the coordinates of portfolio $K(T_0)$,
5. increase the TEV by $h$ and start over until $T_0 < T^{\max}$.

# Economic/Financial implications

1. the RBF contains Pareto-efficient portfolios in terms of TEV-VaR;

# Economic/Financial implications

1. the RBF contains Pareto-efficient portfolios in terms of TEV-VaR;
2. augmenting the TEV/reducing the VaR: the efficiency loss diminishes;

# Economic/Financial implications

1. the RBF contains Pareto-efficient portfolios in terms of TEV-VaR;
2. augmenting the TEV/reducing the VaR: the efficiency loss diminishes;
3. the RBF contains portfolios where $\dfrac{\partial \mathrm{VaR}}{\partial \mathrm{TEV}} > 0$: this can happen when managers aim to increase the expected return;
4. optimal VaR-TEV efficient portfolio choice
   - $U_{\mathrm{MAN}}(\underset{(+)}{\mu_K - \mu_B}, \underset{(-)}{T_0}) \longrightarrow$ selection of risky portfolios (returns higher than the benchmark),
   - $U_{\mathrm{INV}}(\underset{(+)}{\mu_K}, \underset{(-)}{V_0}) \longrightarrow$ risks reduction.

# Economic/Financial implications

1. the RBF contains Pareto-efficient portfolios in terms of TEV-VaR;
2. augmenting the TEV/reducing the VaR: the efficiency loss diminishes;
3. the RBF contains portfolios where $\dfrac{\partial \text{VaR}}{\partial \text{TEV}} > 0$: this can happen when managers aim to increase the expected return;
4. optimal VaR-TEV efficient portfolio choice
   - $U_{\text{MAN}}(\underset{(+)}{\mu_K - \mu_B}, \underset{(-)}{T_0}) \longrightarrow$ selection of risky portfolios (returns higher than the benchmark),
   - $U_{\text{INV}}(\underset{(+)}{\mu_K}, \underset{(-)}{V_0}) \longrightarrow$ risks reduction.
5. empirical analysis: work in progress.

# Thank you. Bye Bye